The category of a partitioned fan

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Polyhedral fans

Polyhedral fans play an important role in *toric geometry*, the theory of *polytopes* and τ*-tilting theory*. A *polyhedral cone* in $\mathbb{R}^n$ is the non-negative linear span of linearly independent vectors. A *fan* Σ in $\mathbb{R}^n$ is a collection of such polyhedral cones in $\mathbb{R}^n$ satisfying the following:
(1) Each face of a cone in Σ is also contained in Σ.
(2) The intersection of two cones in Σ is a face of each of the two cones.

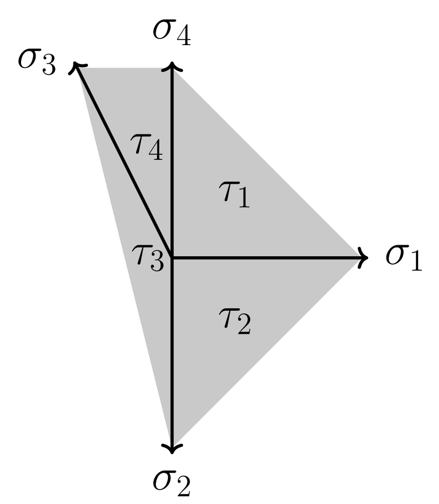


Figure 1. Fan $\Sigma_{\mathbb{F}_a}$ of a Hirzebruch surface $\mathbb{F}_a$

For a cone $\sigma \in \Sigma$, define $\operatorname{star}(\sigma) = \{\tau \in \Sigma : \sigma \subseteq \tau\}$ and the orthogonal projection $\pi_\sigma : \mathbb{R}^n \to \operatorname{span}\{\sigma\}^\perp$.

An admissible partition of the fan

Potential identifications of a cone $\sigma \in \Sigma$:

$$\mathcal{E}_\sigma := \{\kappa \in \Sigma : \operatorname{span}\{\sigma\}^\perp = \operatorname{span}\{\kappa\}^\perp \text{ and } \pi_\sigma(\operatorname{star}(\sigma)) = \pi_\kappa(\operatorname{star}(\kappa))\}.$$

Partitioning the sets $\mathcal{E}_\sigma$ into *actual identifications* $\to$ Partition $\mathfrak{P}$ of Σ. Such a partition $\mathfrak{P}$ is called *admissible* if whenever $\sigma_1 \sim \sigma_2$ are such that $\pi_{\sigma_1}(\tau_1) = \pi_{\sigma_2}(\tau_2)$ for some $\tau_1 \in \operatorname{star}(\sigma_1)$ and $\tau_2 \in \operatorname{star}(\sigma_2)$, then $\tau_1 \sim \tau_2$.

Lemma. Admissible partitions exist.

The category

Given a fan Σ and an admissible partition of its cones $\mathfrak{P}$, define the *category of the partitioned fan*, denoted by $\mathfrak{C}(\Sigma, \mathfrak{P})$, as follows:
(1) Objects of $\mathfrak{C}(\Sigma, \mathfrak{P})$ are equivalence classes $[\sigma]$ of the partition $\mathfrak{P}$.
(2) $\operatorname{Hom}_{\mathfrak{C}(\Sigma,\mathfrak{P})}([\sigma], [\tau])$ consists of equivalence classes of objects in

$$\bigcup_{\sigma' \in [\sigma], \tau' \in [\tau]} \operatorname{Hom}_\Sigma(\sigma', \tau')$$

under the relation $f_{\sigma_1\tau_1} \sim f_{\sigma_2\tau_2}$ if and only if $\pi_{\sigma_1}(\tau_1) = \pi_{\sigma_2}(\tau_2)$.
Lemma. Composition is well-defined (cf. [5, Lem. 3.9, 3.10]).

Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be partitions of a fan Σ. We say that $\mathfrak{P}_1$ is a *finer* partition than $\mathfrak{P}_2$ if

$$\sigma \sim_{\mathfrak{P}_1} \tau \implies \sigma \sim_{\mathfrak{P}_2} \tau$$

for $\sigma, \tau \in \Sigma$. In this case, we write $P_1 \leq P_2$ and say P_2 is *coarser* than P_1. Denote by $\operatorname{APart}(\Sigma)$ the set of all admissible partitions of a fan Σ.
Theorem. The partially ordered set $\operatorname{APart}(\Sigma)$ is a complete lattice.

Cubical categories

The *standard k-cube category* $\mathfrak{I}^k$ is the poset category on subsets of $\{1, \dots, k\}$ under inclusion. For any morphism $(A \xrightarrow{f} B)$ in some category $\mathfrak{C}$, the *factorisation category* $\operatorname{Faq}(f)$ is the category whose objects are factorisations $A \xrightarrow{g} C \xrightarrow{h} B$ such that $h \circ g = f$ and whose morphisms are morphisms $\phi : C_1 \to C_2$ such that $\phi \circ g_1 = g_2$ and $h_1 = h_2 \circ \phi$. Given an object $(A \xrightarrow{g} C \xrightarrow{h} B)$ in $\operatorname{Faq}(f)$, we call g a *first factor of* f if g is irreducible in $\mathfrak{C}$ and h a *last factor of* f if h is irreducible in $\mathfrak{C}$.

A *cubical category*, introduced by Igusa [4], is a small category $\mathfrak{C}$ with the following properties:
(1) Every morphism $f : A \to B$ has a *rank* $\operatorname{rk}(f) \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{rk}(g \circ f) = \operatorname{rk}(f) + \operatorname{rk}(g)$.
(2) If $\operatorname{rk}(f) = k$ then $\operatorname{Faq}(f) \cong \mathfrak{I}^k$.
(3) The forgetful functor $\operatorname{Faq}(f) \to \mathfrak{C}$ taking $A \xrightarrow{g} C \xrightarrow{h} B$ to C is an embedding.
(4) Every morphism of rank k is determined by its k first factors.
(5) Every morphism of rank k is determined by its k last factors.

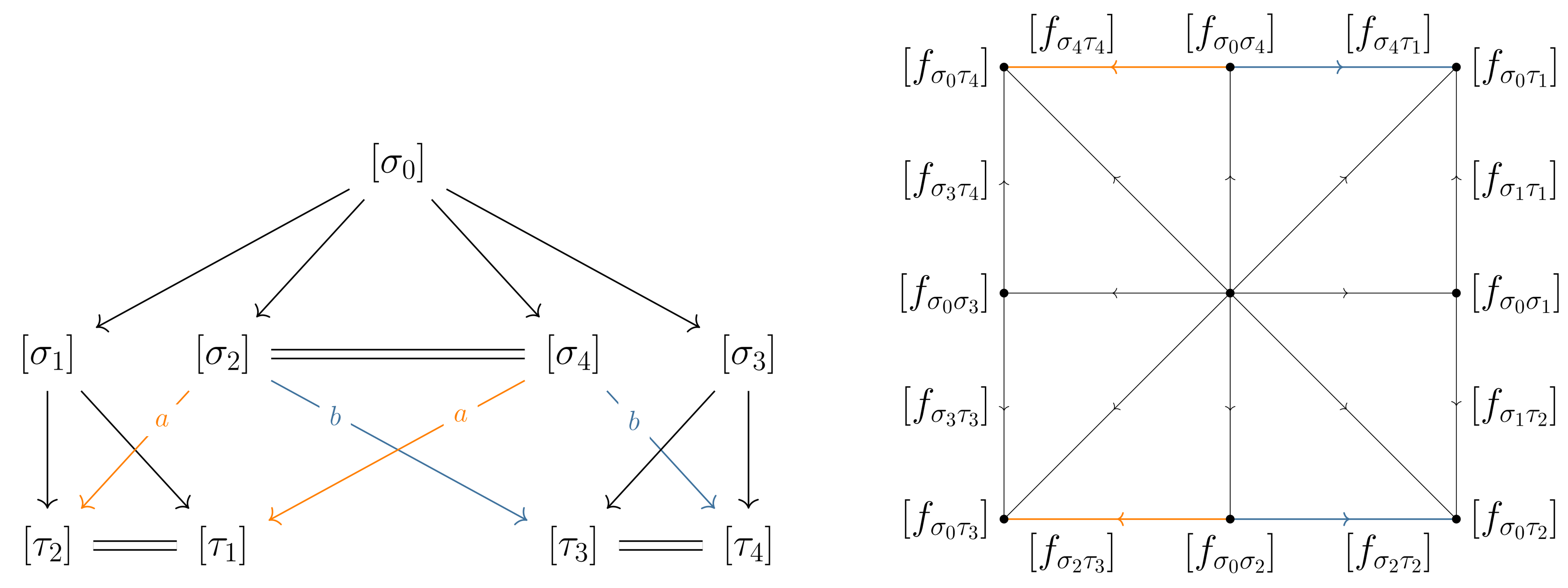
Theorem. The category $\mathfrak{C}(\Sigma, \mathfrak{P})$ of a fan with an admissible partition is cubical.

Proposition. [4] If there exists a faithful functor from $\mathfrak{C}(\Sigma, \mathfrak{P})$ to some group and the category satisfies the "pairwise compatibility of last factors" then its classifying space is a $K(\pi, 1)$ space.

An example

Consider the fan $\Sigma_{\mathbb{F}_a}$ from Figure 1. with the identification $\sigma_2 \sim \sigma_4$. Because $\pi_{\sigma_2}(\tau_2) = \pi_{\sigma_4}(\tau_1)$ and $\pi_{\sigma_2}(\tau_3) = \pi_{\sigma_4}(\tau_4)$ we must also identify $\tau_1 \sim \tau_2$ and $\tau_3 \sim \tau_4$ to make the partition admissible. We obtain the following partition:

$$\mathfrak{P} = \{\{\sigma_0\}, \{\sigma_1\}, \{\sigma_3\}, \{\sigma_2, \sigma_4\}, \{\tau_1, \tau_2\}, \{\tau_3, \tau_4\}\}.$$



(a) The category $\mathfrak{C}(\Sigma_{\mathbb{F}_a}, \mathfrak{P})$

(b) The 2-cell of the CW-complex $\mathcal{B}\mathfrak{C}(\Sigma_{\mathbb{F}_a}, \mathfrak{P})$

Remark. We may also identify all rank 2 cones.

Classifying space

The *classifying space* $\mathcal{B}\mathfrak{C}$ of a category $\mathfrak{C}$ is the geometric realisation of the *simplicial nerve* of the category. 0-simplices correspond to the objects of $\mathfrak{C}$ and the non-degenerate k-simplices correspond to the chains of composable non-identity morphisms $(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_k} X_k)$ in $\mathfrak{C}$.

Theorem. The classifying space $\mathcal{B}\mathfrak{C}(\Sigma, \mathfrak{P})$ is a n-dimensional CW-complex having one cell $e([\sigma])$ of dimension $k = n - \dim(\sigma)$ for each equivalence class $[\sigma] \in \mathfrak{P}$. The k-cell $e([\sigma])$ is the union of the factorisation cubes of the morphisms $[f_{\sigma\tau}]$, where τ is an n-dimensional cone in $\operatorname{star}(\sigma)$.

Picture group

A *weak fan poset* is a pair $(\Sigma, \mathcal{P})$ where Σ is a finite complete fan in $\mathbb{R}^n$ and $\mathcal{P}$ is a poset on Σ^n such that (1) for every interval I of $\mathcal{P}$, the union of all maximal cones in I is strongly-connected and (2) for every cone $\sigma \in \Sigma$, the set of maximal cones containing σ is an interval in $\mathcal{P}$, which we denote by $[\sigma^-, \sigma^+]$.

Definition Let $(\Sigma, \mathfrak{P}, \mathcal{P})$ be a partitioned fan poset. The *picture group* $G(\Sigma, \mathfrak{P}, \mathcal{P})$ has generators $\{X_{[\sigma]} : \sigma \in \Sigma^{n-1}\}$ and the following sets of relations:

- $X_{[\sigma_1]} \dots X_{[\sigma_k]} = X_{[\sigma'_1]} \dots X_{[\sigma'_\ell]}$, whenever $(\sigma_1, \dots, \sigma_k)$ and $(\sigma'_1, \dots, \sigma'_\ell)$ are two distinct sequences of codimension 1 cones labelling the arrows of some interval $[\tau_1, \tau_2]$. Denote this element by $X_{[\tau_1, \tau_2]}$.
- $X_{[\sigma_1^-, \kappa_1^-]} = X_{[\sigma_2^-, \kappa_2^-]}$, whenever $[f_{\sigma_1\kappa_1}] = [f_{\sigma_2\kappa_2}]$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$.

Theorem. The functor $\mathfrak{C}(\Sigma, \mathfrak{P}) \to G(\Sigma, \mathfrak{P}, \mathcal{P})$ sending $[f_{\sigma\kappa}] \mapsto [\sigma^-, \kappa^-]$ is faithful when Σ is a rank 2 fan and $\mathcal{P}$ does not annihilate any generators.
(1) If there are no 3 pairwise compatible rank 1 morphisms, $\mathcal{B}\mathfrak{C}(\Sigma, \mathfrak{P})$ is a $K(\pi, 1)$.
(2) If $\mathfrak{P}$ identifies all maximal cones, then π is the picture group.

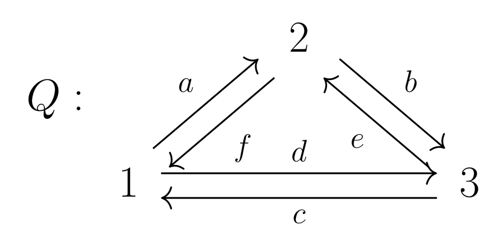
Relationships within the lattice

Theorem. If $\mathfrak{P}_1 \leq \mathfrak{P}_2$ are admissible partitions, then the following hold:
(1) There is a faithful surjective-on-objects functor $F : \mathfrak{C}(\Sigma, \mathfrak{P}_1) \to \mathfrak{C}(\Sigma, \mathfrak{P}_2)$.
(2) The classifying space $\mathcal{B}\mathfrak{C}(\Sigma, \mathfrak{P}_2)$ is a quotient of $\mathcal{B}\mathfrak{C}(\Sigma, \mathfrak{P}_1)$.
(3) The picture group $G(\Sigma, \mathfrak{P}_2, \mathcal{P})$ is a quotient of $G(\Sigma, \mathfrak{P}_1, \mathcal{P})$.

Corollary. If $\mathfrak{C}(\Sigma, \mathfrak{P}_2)$ admits a faithful functor to some group, then so does the category of any finer partition.

Brauer cycle algebra

Let $A = KQ/I$ be the *Brauer cycle algebra* of rank 3, which is given by



$$I = \langle ab, bc, ca, de, ef, fd, af - dc, be - fa, cd - eb \rangle.$$

Theorem. The classifying space $\mathcal{B}\mathfrak{C}(A)$ of the τ-cluster morphism category is a $K(\pi, 1)$ space for the picture group $G(A)$.
Proof. (1) The g-fan $\Sigma(A)$ is a hyperplane arrangement.
(2) There exists a faithful functor from $\mathfrak{C}(\Sigma(A), \mathfrak{P}_{\text{flat}}) \to G(\Sigma(A), \mathfrak{P}_{\text{flat}}, \mathcal{P})$, where $\mathfrak{P}_{\text{flat}}$ is the maximal partition and $\mathcal{P}$ the poset of regions.
(3) The partition $\mathfrak{P}_{\text{WAC}}$ giving rise to the τ-cluster morphism category is finer than $\mathfrak{P}_{\text{flat}}$, hence there exists a faithful functor to the same group given by composition.
(4) The pairwise compatibility of last factors is satisfied by [1] hence $\mathcal{B}\mathfrak{C}(A)$ is a $K(\pi, 1)$ and the picture group is isomorphic to the fundamental group by [3].

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