A lattice of categories of an algebra

A: fd algebra over k, $\mathrm{mod}\,A$: fg (right) modules of A,
τ: Auslander–Reiten translation

§1 Background

Def [AIR] A pair $(M,P) \in \mathrm{mod}\,A \times \mathrm{proj}\,A$ is τ-rigid if $\mathrm{Hom}(M, \tau M) = 0$ and $\mathrm{Hom}(P, M) = 0$. If additionally $|M| + |P| = |A|$, we call it τ-tilting.

Thm [AIR] $\exists$ poset isomorphisms between:
- τ-tilting pairs
- functorially finite torsion classes of $\mathrm{mod}\,A$.
- 2-term silting objects in $K^b(\mathrm{proj}\,A)$

Def/Prop [Asai] For every cover relation in the posets above, there exists a "brick-label"

e.g. $\quad \mathcal{T}_1 \overset{B}{\lessdot} \mathcal{T}_2 \quad$ if $\quad \mathcal{T}_1^{\perp} \cap \mathcal{T}_2 = \mathrm{Filt}\{B\}$

Def [ITW, HI] The picture group $G(A)$ has generators $\{X_B : B \in \mathrm{brick}A\} \cup \{g_{\mathcal{T}} : \mathcal{T} \in \text{f-tors}\,A\}$ and relations

$$g_{\mathcal{T}_2} = X_B\, g_{\mathcal{T}_1} \qquad \text{for} \qquad \mathcal{T}_1 \overset{B}{\lessdot} \mathcal{T}_2$$

and $g_0 = e_0$

§2 The τ-clustermorphism category

Def [DIJ] Let $\bigoplus_{i=1}^{|A|} P(i)^{b_i} \to \bigoplus_{i=1}^{|A|} P(i)^{a_i} \to M \to 0$ be a min. proj. presentation then the g-vector is

$$g^M := (a_1 - b_1, \ldots, a_{|A|} - b_{|A|}) \in \mathbb{R}^{|A|}$$

The g-vector fan $\Sigma(A) := \{ \mathcal{C}_{(M,P)} : (M,P) \text{ is } \tau\text{-rigid}\}$ where
$\mathcal{C}_{(M,P)} := \mathrm{span}_{\geq 0}\{ g^{M_1}, \ldots, g^{M_r}, -g^{P_{r+1}}, \ldots, -g^{P_t}\}$.

Def [STTW] The τ-cluster morphism category $\mathfrak{C}(A)$ has
- objects: equiv. classes under
$$\mathcal{C}_{(M_1,P_1)} \sim \mathcal{C}_{(M_2,P_2)} \text{ whenever } M_1^{\perp} \cap {}^{\perp}\tau M_1 \cap P_1^{\perp} = M_2^{\perp} \cap {}^{\perp}\tau M_2 \cap P_2^{\perp}.$$

These identifications imply ?
- $\mathrm{span}\{\mathcal{C}_{(M_1,P_1)}\} \overset{?}{=} \mathrm{span}\{\mathcal{C}_{(M_2,P_2)}\}$

Define $\pi_{(M,P)} : \mathbb{R}^{|A|} \to \mathrm{span}\{\mathcal{C}_{(M,P)}\}^{\perp}$ and $\mathrm{star}\,\mathcal{C}_{(M,P)} = \{\mathcal{C}_{(N,Q)} : \mathcal{C}_{(M,P)} \subseteq \mathcal{C}_{(N,Q)}\}$

- morphisms of $\mathfrak{C}(A)$ are equiv. classes

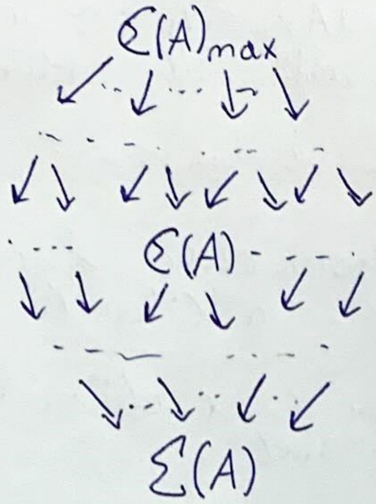
$$f_{\mathcal{C}(N_1,Q_1)\mathcal{C}(M_1,P_1)} \sim f_{\mathcal{C}(N_2,Q_2)\mathcal{C}(M_2,P_2)}$$

whenever $\pi_{(N_1,Q_1)}(\mathcal{C}_{(M_1,P_1)}) = \pi_{(N_2,Q_2)}(\mathcal{C}_{(M_2,P_2)})$.

§3 The lattice of categories

<u>Idea:</u> Allow more general identifications of g-vector cones following the geometric constraints above.

<u>Thm</u> This leads to a lattice of categories



$\exists$ faithful functor

<u>Goal:</u> Show $B\mathcal{E}(A)$ is a $K(\mathcal{G}(A),1)$.

[Igusa] $\exists$ 2 sufficient conditions

- "pairwise compatibility condition"
- existence of a faithful functor $\mathcal{E}(A) \to G$.

<u>Thm</u> If $\Sigma(A)$ is a hyperplane arrangement, then there exists a faithful functor $\mathcal{E}(A) \to G$.

$$\begin{array}{cccccc} \text{"Proof" dimension vectors of bricks} & \mathcal{E}(A)_{max} & \xrightarrow{\text{faithful}} & \mathcal{G}(A)_{max} & \xrightarrow[\text{hom.}]{\text{gp.}} & \mathbb{Z}[[\mathbb{R}^{|A|}]] \\ & \uparrow & \circlearrowleft & \nearrow & & \\ \text{bricks} & \mathcal{E}(A) & \hookrightarrow & \mathcal{G}(A) & \xrightarrow[\text{hom.}]{\text{gp.}} & \text{Hall algebra} \end{array}$$