The category of a

Partitioned fans, hyperplane arrangements and $K(\pi, 1)$ spaces

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Polyhedral fans

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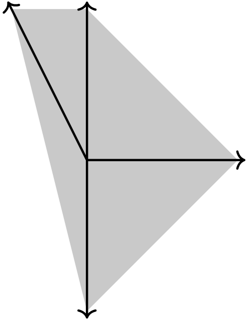
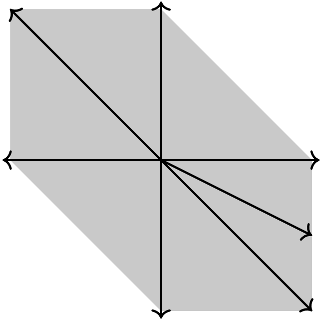
- *finite* if $|\Sigma| < \infty$.
- *complete* if $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$.
- *simplicial* if $v_1, \ldots, v_s$ are linearly independent for each cone.

Examples of fans



(a) Hirzebruch surface $\mathbb{F}_a$

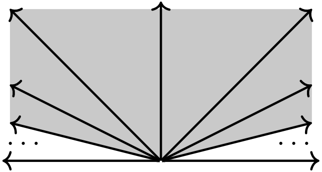
Toric varieties



(b) g-vector fan of an algebra

f.d. algebra

[Demonet–Iyama–Jasso '18]



Abelian cat

[Broomhead–Pauksztello–Ploog–Woolf '23]

(c) Abelian category of coherent sheaves on the projective line

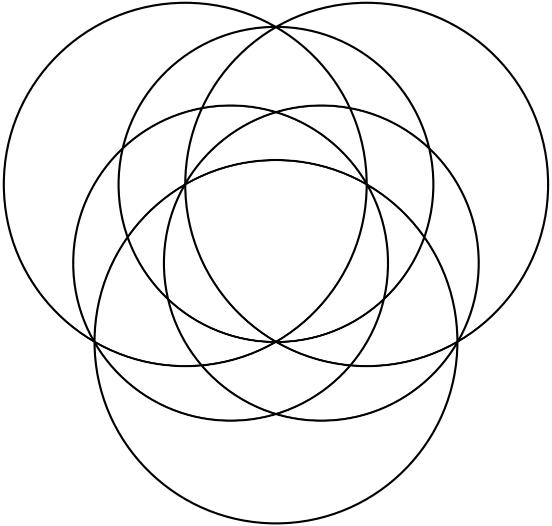
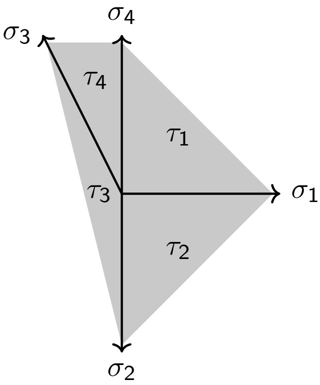


Figure: Hyperplane arrangement

central
$\cap H = \emptyset$

Partition of a fan



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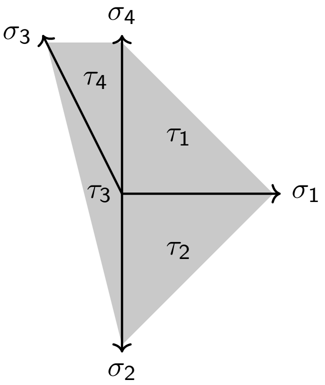
Partition $\mathfrak{P}$ of the cones of the fan Σ: $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ only if:

Partition of a fan



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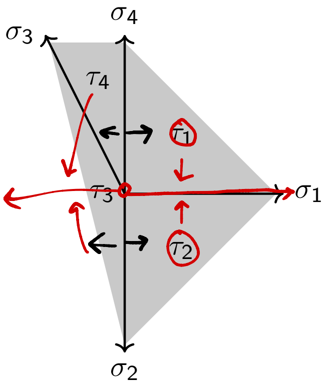

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2. $p_{\kappa_1^\perp}(\text{star}(\kappa_1)) = p_{\kappa_2^\perp}(\text{star}(\kappa_2))$.

$p_{\kappa_1^\perp} : \mathbb{R}^n \to \text{span}\{\kappa_1\}^\perp$

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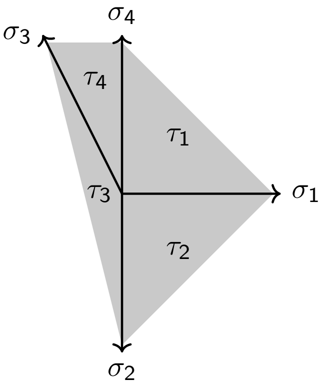

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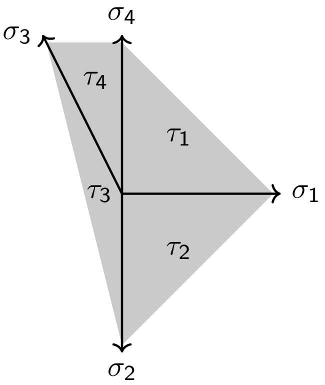

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Example:

- σ_1 and σ_3 may not be identified.
- σ_2 and σ_4 may be identified but then τ_1 and τ_2 as well as τ_3 and τ_4 have to be identified.

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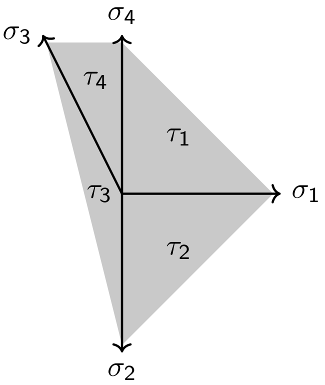

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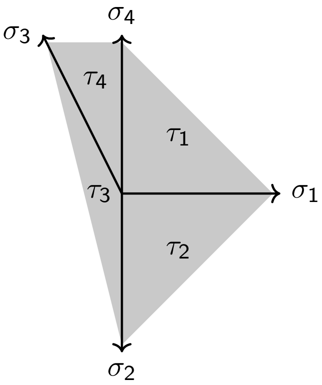
Definition (The category of a partitioned fan)

Objects: equivalence classes of cones in $\mathfrak{P}$.
Morphisms: $f_{[\kappa],[\lambda]} = \bigcup \text{Hom}_\Sigma(\kappa_i, \lambda_j)$ with the relation $f_{\kappa_1\lambda_1} \sim f_{\kappa_2\lambda_2}$ whenever $p_{\kappa_1^\perp}(\lambda_1) = p_{\kappa_2^\perp}(\lambda_2)$.

Category of a partitioned fan



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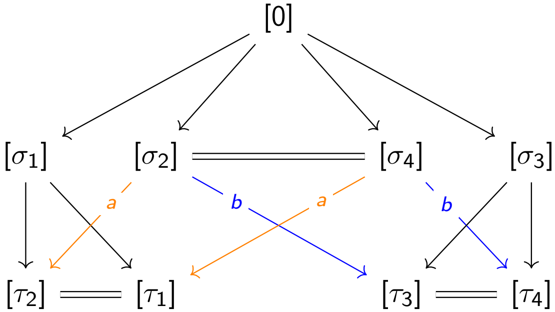

(b) The category of the partitioned fan $(\Sigma, \mathfrak{P}_1)$.

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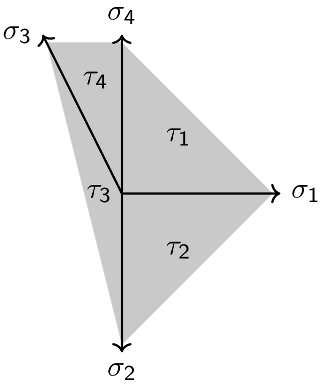

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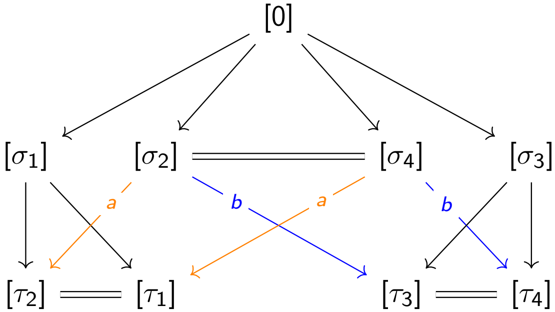

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τ-cluster morphism category

see also [Igusa-Todorov, 2017], [Buan-Marsh, 2021], [Buan-Hanson, 2023], [Schroll-Tattar-Treffinger-Williams, 2023]

Some natural partitions of fans

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   - cones are $\mathcal{C}_{(M,P)}$ where (M, P) is τ-rigid.

A : fd algebra

$M \in \operatorname{mod} A$

$P \in \operatorname{proj} A$

τ_A: Auslander-Reiten translation

$\operatorname{Hom}_A(M, \tau_A M) = 0$

$\operatorname{Hom}_A(P, M) = 0$

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   - identification $\mathcal{C}_{(M_1,P_1)} \sim \mathcal{C}_{(M_2,P_2)}$ whenever $M_1^{\perp} \cap {}^{\perp}\tau M_1 \cap P_1^{\perp} = M_2^{\perp} \cap {}^{\perp}\tau M_2 \cap P_2^{\perp}$.

wide sub.

$M_1^{\perp} = \{X \in \text{mod}\,A \mid \text{Hom}_A(M_1, X) = 0\}$

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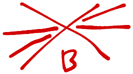
   - Flats are intersections of hyperplanes.
   - Shards are parts of hyperplanes determined by a choice of base regions.



Fan poset

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Definition (Reading, 2005)

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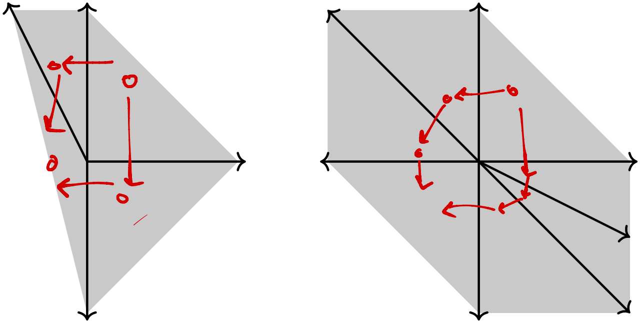
1. For every interval I of $\mathcal{P}$, the union of all maximal cones in I is a polyhedral cone (resp. simply-connected);
2. For every cone $\sigma \in \Sigma$, the set of maximal cones, $\text{star}(\sigma)^n$, containing σ is an interval in $\mathcal{P}$, which we denote by $[\sigma^-, \sigma^+]$ and call a *facial interval*.

Fan poset

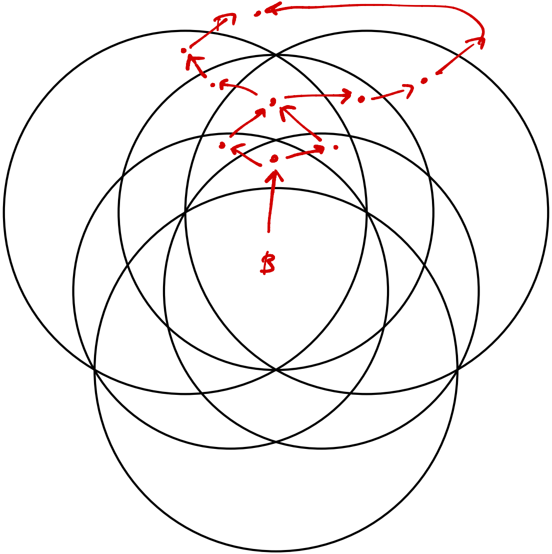


(a) Hirzebruch surface $\mathbb{F}_a$

weak fan poset

(b) g-vector fan of an algebra

$\mathcal{C}_{(M,P)} < \mathcal{C}_{(N,Q)}$ whenever $\operatorname{Gen} M \subset \operatorname{Gen} N$ [Adachi-Iyama-Reiten '14]



Poset of regions (= max. cones) [Edelman 1980s]

choose base region B

orienting the adjacency graph away from B

Figure: Hyperplane arrangement

Incidence group

Definition

The incidence group $G(\Sigma, \mathfrak{P}, \mathcal{P})$ may be presented with the set of generators $\{X_{[\sigma]} : \sigma \in \Sigma^{n-1}\} \cup \{g_\tau : \tau \in \Sigma^n\}$ and a relation

$$g_{\tau_1} = X_{[\sigma]} g_{\tau_2}$$

if there is an arrow $\tau_1 \xrightarrow{\sigma} \tau_2$ in Hasse($\mathcal{P}$) labelled by σ and the relation $g_{0^-} = e$.

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See also [Igusa-Todorov-Weyman, 2016], [Hanson-Igusa, 2021]

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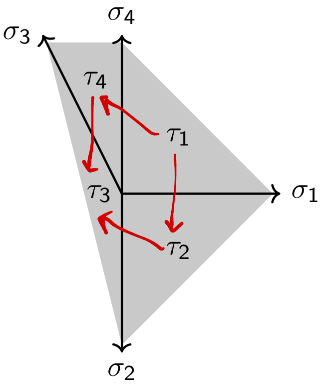

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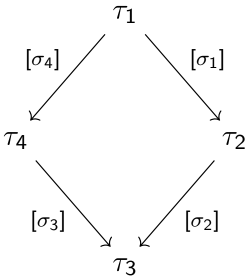

(b) Hasse diagram of the fan poset

$\langle g_{\tau_i}, X_{[\sigma_i]} \mid g_{\tau_1} = X_{[\sigma_4]} g_{\tau_4},\ g_{\tau_1} = X_{[\sigma_1]} g_{\tau_2},\ g_{\tau_3} = e,\ g_{\tau_2} = X_{[\sigma_2]},\ g_{\tau_4} = X_{[\sigma_3]} \rangle$

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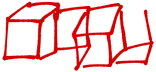
Theorem

The classifying space of the category of a partitioned fan is a cube complex.

0-simplices : identity morph.

k- simplices : chains of k composable non-id. morph.



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Sufficient conditions [Gromov, 1987][Igusa, 2014]

1. There exists a faithful functor to some group G.

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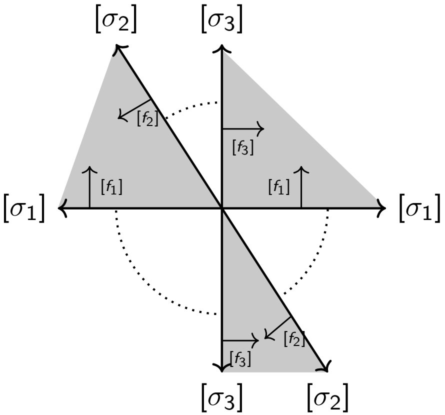
Sufficient conditions [Gromov, 1987][Igusa, 2014]

1. There exists a faithful functor to some group G.
2. The category of a partitioned fan satisfies the "pairwise compatibility property".

⇒ universal covering is a cube complex

$K(\pi,1)$ ⇐ contractible ⇐ a cube complex is CAT(0) iff every vertex link is a flag simp. compl.

Results in rank 2



(a) Three pairs of compatible morphisms.

Lemma

$\mathfrak{C}(\Sigma, \mathfrak{P})$ *satisfies the pairwise compatibility property if and only there is no set of three pairwise compatible rank 1 morphisms.*

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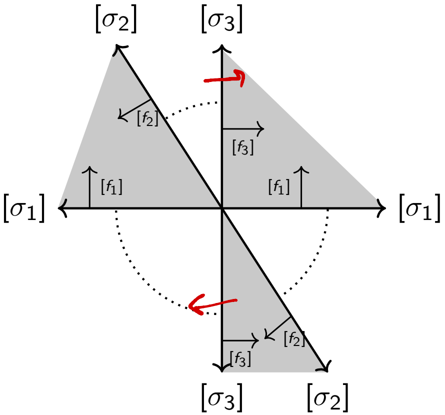

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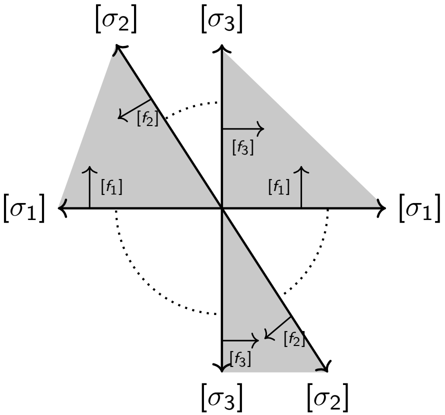

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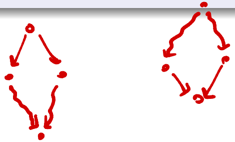
If the partition $\mathfrak{P}$ identifies all maximal cones, the fundamental group is the incidence group.

General results

Proposition

Let $(\Sigma, \mathfrak{P}, \mathcal{P})$ *be a non-degenerate partitioned fan poset. If* $\mathcal{P}$ *is a polygonal lattice and* $\mathfrak{P}$ *identifies all maximal cones of* Σ*, then the fundamental group of the classifying space of* $\mathfrak{C}(\Sigma, \mathfrak{P})$ *is isomorphic to the incidence group.*



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Theorem

For each simplicial fan, the categories of a partitioned fan form a complete lattice defined by refinement of the partitions.

$\mathfrak{P}_1 \underset{\text{finer}}{<} \mathfrak{P}_2 \iff x \sim_{\mathfrak{P}_1} y \implies x \sim_{\mathfrak{P}_2} y$

General results

Proposition

Let $(\Sigma, \mathfrak{P}, \mathcal{P})$ be a non-degenerate partitioned fan poset. If $\mathcal{P}$ is a polygonal lattice and $\mathfrak{P}$ identifies all maximal cones of Σ, then the fundamental group of the classifying space of $\mathfrak{C}(\Sigma, \mathfrak{P})$ is isomorphic to the incidence group.

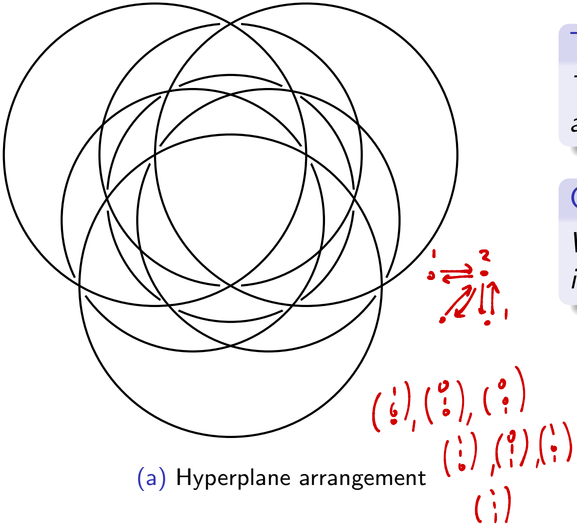
Theorem

For each simplicial fan, the categories of a partitioned fan form a complete lattice defined by refinement of the partitions.

Proposition

Let $\mathfrak{P}_1$ be a finer partition than $\mathfrak{P}_2$. If there exists a faithful group functor for $\mathfrak{C}(\Sigma, \mathfrak{P}_2)$ then there exists a faithful group functor for $\mathfrak{C}(\Sigma, \mathfrak{P}_1)$.

$K(\pi, 1)$ for a partitioned hyperplane arrangement



(a) Hyperplane arrangement

Theorem

The category of the flat-partitioned hyperplane arrangement admits a faithful group functor.

Corollary

With the shard-partition, the classifying space is a $K(\pi, 1)$ space for the incidence group.

- Faithful functor is inherited from the coarser flat-partition.
- Pairwise compatibility follows from [Barnard-Hanson, 2022] and [Mizuno, 2022]

Thank you!