The category of a partitioned fan

Maximilian Kaipel

University of Cologne

mkaipel@uni-koeln.de

Λ: finite-dimensional algebra with $|\operatorname{brick}\Lambda| < \infty$.

Λ: finite-dimensional algebra with $|\operatorname{brick}\Lambda| < \infty$.

Definition (1),(2)

Picture group $G(\Lambda)$

Generators: X_S, where $S \in \operatorname{brick}\Lambda$

Relations: $X_{S_1} \dots X_{S_k} = X_{S'_1} \dots X_{S'_\ell}$

if $(S_1, \dots, S_k)$ and $(S'_1, \dots, S'_\ell)$ correspond with maximal green sequences.

Λ: finite-dimensional algebra with $|\operatorname{brick}\Lambda| < \infty$.

Definition (1),(2)

Picture group $G(\Lambda)$

Generators: X_S, where $S \in \operatorname{brick}\Lambda$

Relations: $X_{S_1} \dots X_{S_k} = X_{S'_1} \dots X_{S'_\ell}$

if $(S_1, \dots, S_k)$ and $(S'_1, \dots, S'_\ell)$ correspond with maximal green sequences.

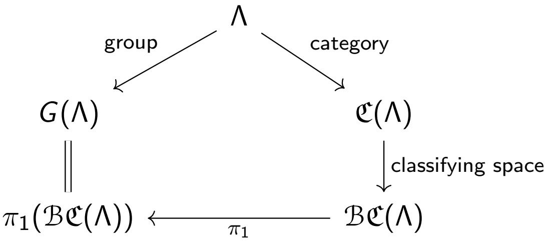
Theorem (3)

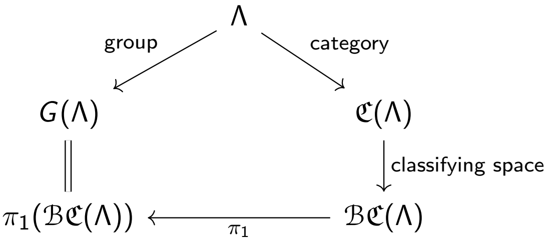
For Λ hereditary, the "positive expressions" for the element $X_{S_1} \dots X_{S_k}$ are in bijection with maximal green sequences.

(1) K. Igusa, G. Todorov and J. Weyman, "Picture groups of finite type and cohomology in type $\mathbb{A}_n$", 2016.
(2) E. J. Hanson and K. Igusa, "τ-cluster morphism categories and picture groups", 2021.
(3) K. Igusa and G. Todorov, "Picture groups and maximal green sequences", 2021.

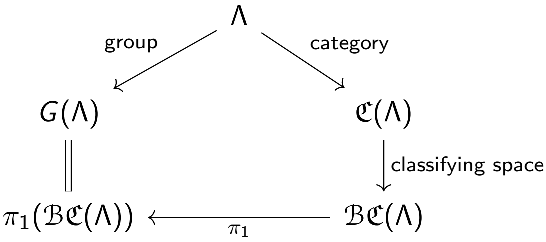
$$\Lambda \xrightarrow{\text{group}} G(\Lambda)$$

$$\begin{array}{ccc} & \Lambda & \\ {\scriptstyle \text{group}} \swarrow & & \searrow {\scriptstyle \text{category}} \\ G(\Lambda) & & \mathfrak{C}(\Lambda) \end{array}$$

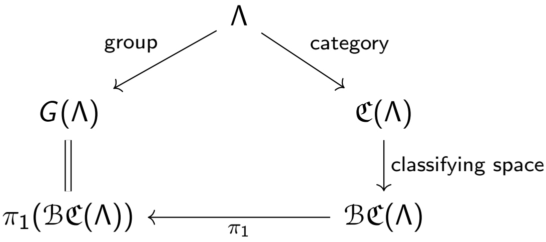

Λ
group
category
$G(\Lambda)$
$\mathfrak{C}(\Lambda)$
classifying space
$\pi_1(\mathcal{B}\mathfrak{C}(\Lambda))$
π_1
$\mathcal{B}\mathfrak{C}(\Lambda)$



$\mathfrak{C}(\Lambda)$ is called the τ-cluster morphism category. (4), (5), (6).



$\mathfrak{C}(\Lambda)$ is called the τ-cluster morphism category. (4), (5), (6).
Objects: wide subcategories - Morphisms: τ-tilting reduction (7)



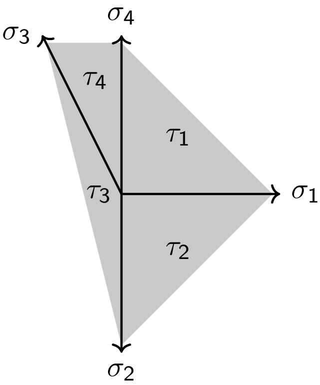
$\mathfrak{C}(\Lambda)$ is called the τ-cluster morphism category. (4), (5), (6).
Objects: wide subcategories - Morphisms: τ-tilting reduction (7)

Early 2023: Geometric construction from g-vector fan. (8)

(4) K. Igusa, G. Todorov, “Signed exceptional sequences and the cluster morphism category”, 2017.
(5) A. B. Buan and B. R. Marsh, “A category of wide subcategories”, 2021.
(6) A. B. Buan and E. J. Hanson, “τ-perpendicular wide subcategories”, 2023.
(7) G. Jasso, “Reduction of τ-tilting modules and torsion pairs”, 2015.
(8) S. Schroll, A. Tattar, H. Treffinger and N. J. Williams, “A geometric perspective on the τ-cluster morphism category”, 2023.

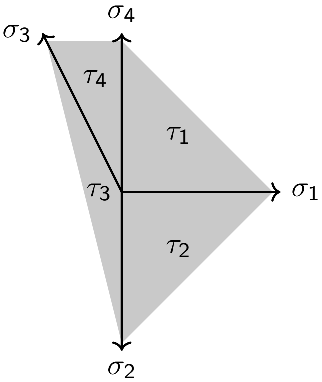
Σ: simplicial polyhedral fan in $\mathbb{R}^n$.

Σ: simplicial polyhedral fan in $\mathbb{R}^n$.



(a) Fan of Hirzebruch surface $\mathbb{F}_a$

Σ: simplicial polyhedral fan in $\mathbb{R}^n$.

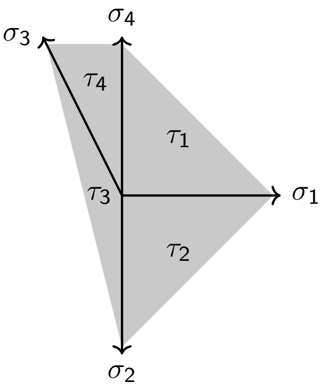


(a) Fan of Hirzebruch surface $\mathbb{F}_a$

Partition $\mathfrak{P}$ of the cones of the fan Σ: $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ only if:

1. $\text{span}\{\kappa_1\}^\perp = \text{span}\{\kappa_2\}^\perp$

Σ: simplicial polyhedral fan in $\mathbb{R}^n$.

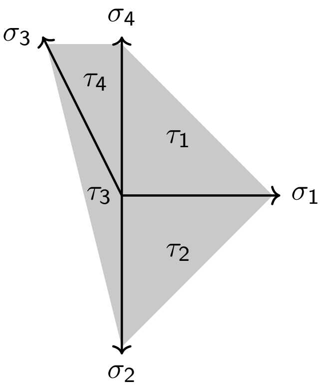


(a) Fan of Hirzebruch surface $\mathbb{F}_a$

Partition $\mathfrak{P}$ of the cones of the fan Σ: $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ only if:

1. $\text{span}\{\kappa_1\}^\perp = \text{span}\{\kappa_2\}^\perp$
2. $p_{\kappa_1^\perp}(\text{star}(\kappa_1)) = p_{\kappa_2^\perp}(\text{star}(\kappa_2))$.

Σ: simplicial polyhedral fan in $\mathbb{R}^n$.



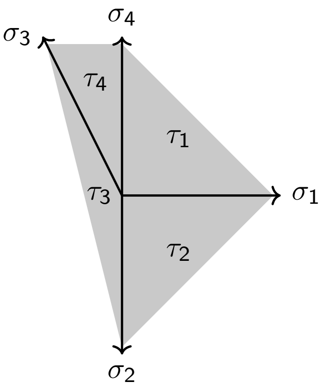
(a) Fan of Hirzebruch surface $\mathbb{F}_a$

Partition $\mathfrak{P}$ of the cones of the fan Σ: $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ only if:

1. $\text{span}\{\kappa_1\}^\perp = \text{span}\{\kappa_2\}^\perp$
2. $p_{\kappa_1^\perp}(\text{star}(\kappa_1)) = p_{\kappa_2^\perp}(\text{star}(\kappa_2))$.

Admissible: If $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ and $p_{\kappa_1^\perp}(\lambda_1) = p_{\kappa_2^\perp}(\lambda_2)$ for some $\lambda_i \in \text{star}(\kappa_i)$, then $\lambda_1 \sim_{\mathfrak{P}} \lambda_2$.

Σ: simplicial polyhedral fan in $\mathbb{R}^n$.



(a) Fan of Hirzebruch surface $\mathbb{F}_a$

Partition $\mathfrak{P}$ of the cones of the fan Σ: $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ only if:

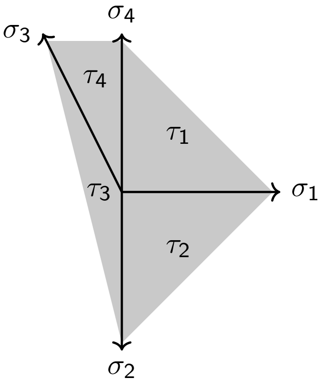
1. $\text{span}\{\kappa_1\}^\perp = \text{span}\{\kappa_2\}^\perp$
2. $p_{\kappa_1^\perp}(\text{star}(\kappa_1)) = p_{\kappa_2^\perp}(\text{star}(\kappa_2))$.

Admissible: If $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ and $p_{\kappa_1^\perp}(\lambda_1) = p_{\kappa_2^\perp}(\lambda_2)$ for some $\lambda_i \in \text{star}(\kappa_i)$, then $\lambda_1 \sim_{\mathfrak{P}} \lambda_2$.

Example:

- σ_1 and σ_3 may not be identified.
- σ_2 and σ_4 may be identified but then τ_1 and τ_2 as well as τ_3 and τ_4 have to be identified.

Σ: simplicial polyhedral fan in $\mathbb{R}^n$.



(a) Fan of Hirzebruch surface $\mathbb{F}_a$

Partition $\mathfrak{P}$ of the cones of the fan Σ: $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ only if:

1. $\text{span}\{\kappa_1\}^\perp = \text{span}\{\kappa_2\}^\perp$
2. $p_{\kappa_1^\perp}(\text{star}(\kappa_1)) = p_{\kappa_2^\perp}(\text{star}(\kappa_2))$.

Admissible: If $\kappa_1 \sim_{\mathfrak{P}} \kappa_2$ and $p_{\kappa_1^\perp}(\lambda_1) = p_{\kappa_2^\perp}(\lambda_2)$ for some $\lambda_i \in \text{star}(\kappa_i)$, then $\lambda_1 \sim_{\mathfrak{P}} \lambda_2$.

Example:

- σ_1 and σ_3 may not be identified.
- σ_2 and σ_4 may be identified but then τ_1 and τ_2 as well as τ_3 and τ_4 have to be identified.

Definition (The category of a partitioned fan)

Objects: equivalence classes of cones in $\mathfrak{P}$.
Morphisms: $f_{[\kappa],[\lambda]} = \bigcup \text{Hom}_\Sigma(\kappa_i, \lambda_j)$ with the relation $f_{\kappa_1\lambda_1} \sim f_{\kappa_2\lambda_2}$ whenever $p_{\kappa_1^\perp}(\lambda_1) = p_{\kappa_2^\perp}(\lambda_2)$.

Theorem

The categories of a partitioned fan define a lattice of categories given by refinement of admissible partitions.

Theorem

The categories of a partitioned fan define a lattice of categories given by refinement of admissible partitions.

The τ-cluster morphism category $\mathfrak{C}(\Lambda)$ is defined from the g-vector fan with identification of cones given by: $\mathcal{C}_{(M_1,P_1)} \sim \mathcal{C}_{(M_2,P_2)}$ if and only if $M_1^{\perp} \cap {}^{\perp}\tau M_1 \cap P_1^{\perp} = M_2^{\perp} \cap {}^{\perp}\tau M_2 \cap P_2^{\perp}$.

Theorem

The categories of a partitioned fan define a lattice of categories given by refinement of admissible partitions.

The τ-cluster morphism category $\mathfrak{C}(\Lambda)$ is defined from the g-vector fan with identification of cones given by: $\mathcal{C}_{(M_1,P_1)} \sim \mathcal{C}_{(M_2,P_2)}$ if and only if $M_1^\perp \cap {}^\perp\tau M_1 \cap P_1^\perp = M_2^\perp \cap {}^\perp\tau M_2 \cap P_2^\perp$.

Question: Is $\mathcal{B}\mathfrak{C}(\Lambda)$ a $K(\pi, 1)$ space for $\pi = G(\Lambda)$, see (2), (4), (9).

Theorem

The categories of a partitioned fan define a lattice of categories given by refinement of admissible partitions.

The τ-cluster morphism category $\mathfrak{C}(\Lambda)$ is defined from the g-vector fan with identification of cones given by: $\mathcal{C}_{(M_1,P_1)} \sim \mathcal{C}_{(M_2,P_2)}$ if and only if $M_1^\perp \cap {}^\perp\tau M_1 \cap P_1^\perp = M_2^\perp \cap {}^\perp\tau M_2 \cap P_2^\perp$.

Question: Is $\mathcal{B}\mathfrak{C}(\Lambda)$ a $K(\pi, 1)$ space for $\pi = G(\Lambda)$, see (2), (4), (9).

Proposition (10)

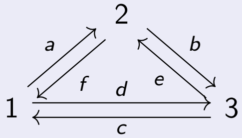
If there exists a faithful functor $F : \mathfrak{C}(\Lambda) \to G$ for some group G and $\mathfrak{C}(\Lambda)$ satisfies the "pairwise compatibility of last factors" then $\mathcal{B}\mathfrak{C}(\Lambda)$ is a $K(\pi, 1)$ space.

(2) E. J. Hanson and K. Igusa, "τ-cluster morphism categories and picture groups", 2021.
(4) K. Igusa, G. Todorov, "Signed exceptional sequences and the cluster morphism category", 2017.
(9) E. J. Hanson and K. Igusa, "Pairwise compatibility for 2-simple minded collections", 2021.
(10) K. Igusa "The category of noncrossing partitions", 2014.

Theorem

Let $\Lambda = KQ/I$ *be the Brauer graph algebra of the 3-cycle, given by*

Q :



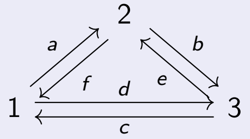
, $I = \langle ab, bc, ca, de, ef, fd, af - dc, be - fa, cd - eb \rangle.$

Then $\mathcal{BC}(\Lambda)$ *is a* $K(\pi, 1)$ *space.*

Theorem

Let $\Lambda = KQ/I$ be the Brauer graph algebra of the 3-cycle, given by

$Q:$



$$I = \langle ab, bc, ca, de, ef, fd, af - dc, be - fa, cd - eb \rangle.$$

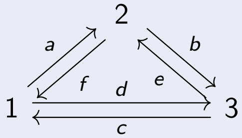
Then $\mathcal{B}\mathfrak{C}(\Lambda)$ is a $K(\pi, 1)$ space.

Proof:

1. $\Sigma(\Lambda)$ is a hyperplane arrangement.
2. There exists a faithful functor $\mathfrak{C}(\Sigma(\Lambda), \mathfrak{P}_{\mathrm{max}}) \to G(\Sigma(\Lambda), \mathfrak{P}_{\mathrm{max}})$.
3. $\mathfrak{C}(\Lambda)$ corresponds to a finer partition, hence we obtain an induced faithful functor $\mathfrak{C}(\Lambda) \to G(\Sigma(\Lambda), \mathfrak{P}_{\mathrm{max}})$.
4. $\mathfrak{C}(\Lambda)$ satisfies the pairwise compatibility of last factors. (11)

Theorem

Let $\Lambda = KQ/I$ be the Brauer graph algebra of the 3-cycle, given by

$Q:$ 

, $\quad I = \langle ab, bc, ca, de, ef, fd, af - dc, be - fa, cd - eb \rangle.$

Then $\mathcal{B}\mathfrak{C}(\Lambda)$ is a $K(\pi, 1)$ space.

Proof:

1. $\Sigma(\Lambda)$ is a hyperplane arrangement.
2. There exists a faithful functor $\mathfrak{C}(\Sigma(\Lambda), \mathfrak{P}_{\mathrm{max}}) \to G(\Sigma(\Lambda), \mathfrak{P}_{\mathrm{max}})$.
3. $\mathfrak{C}(\Lambda)$ corresponds to a finer partition, hence we obtain an induced faithful functor $\mathfrak{C}(\Lambda) \to G(\Sigma(\Lambda), \mathfrak{P}_{\mathrm{max}})$.
4. $\mathfrak{C}(\Lambda)$ satisfies the pairwise compatibility of last factors. (11)

(11) E. Barnard and E. J. Hanson, "Pairwise compatibility for 2-simple minded collections II: Preprojective algebras and semibrick pairs of full rank", 2022.

Thank you!