Torsion lattices and the τ-cluster morphism category

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University of Cologne

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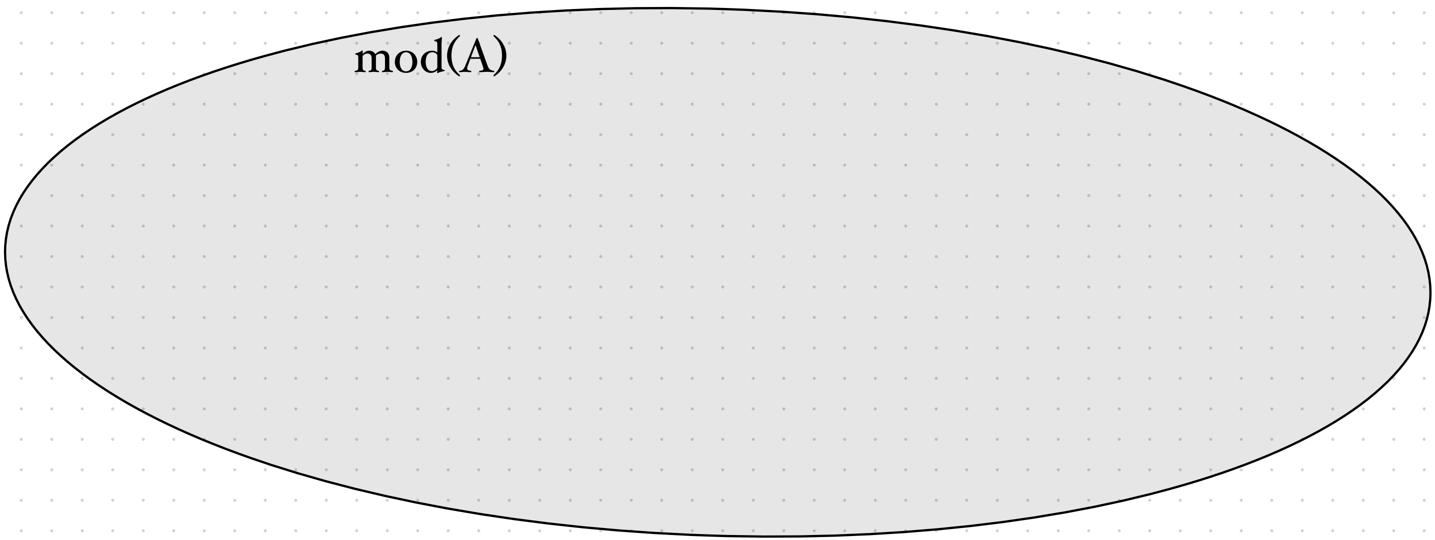
The module category mod(A) of a finite-dimensional algebra A



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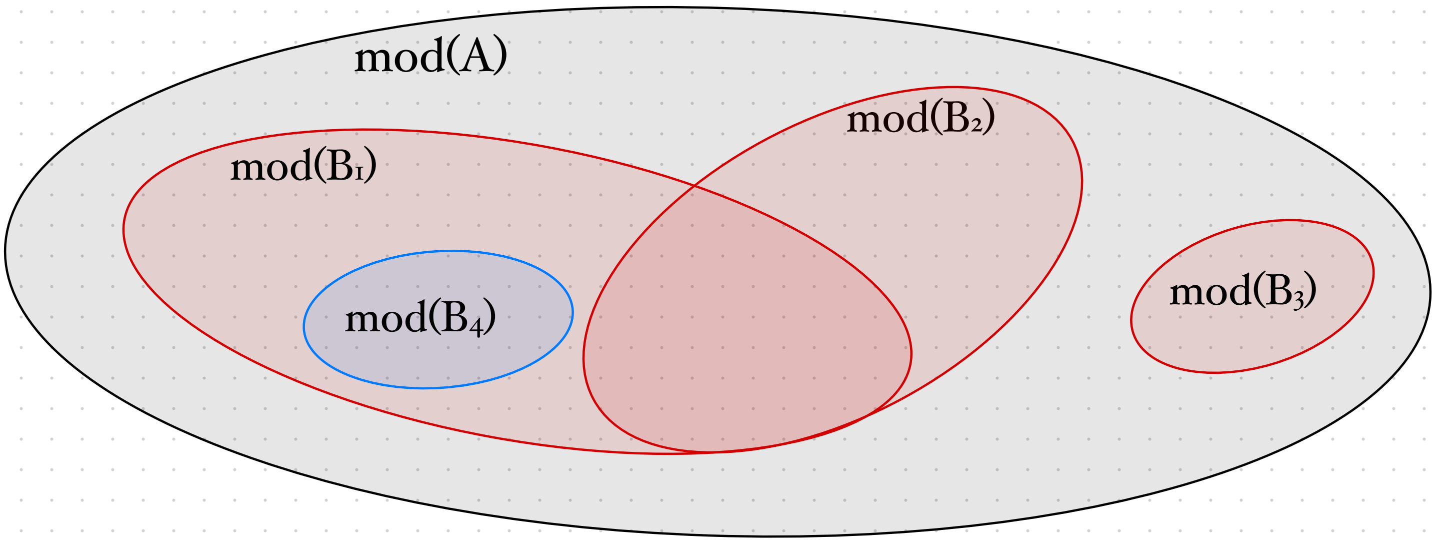

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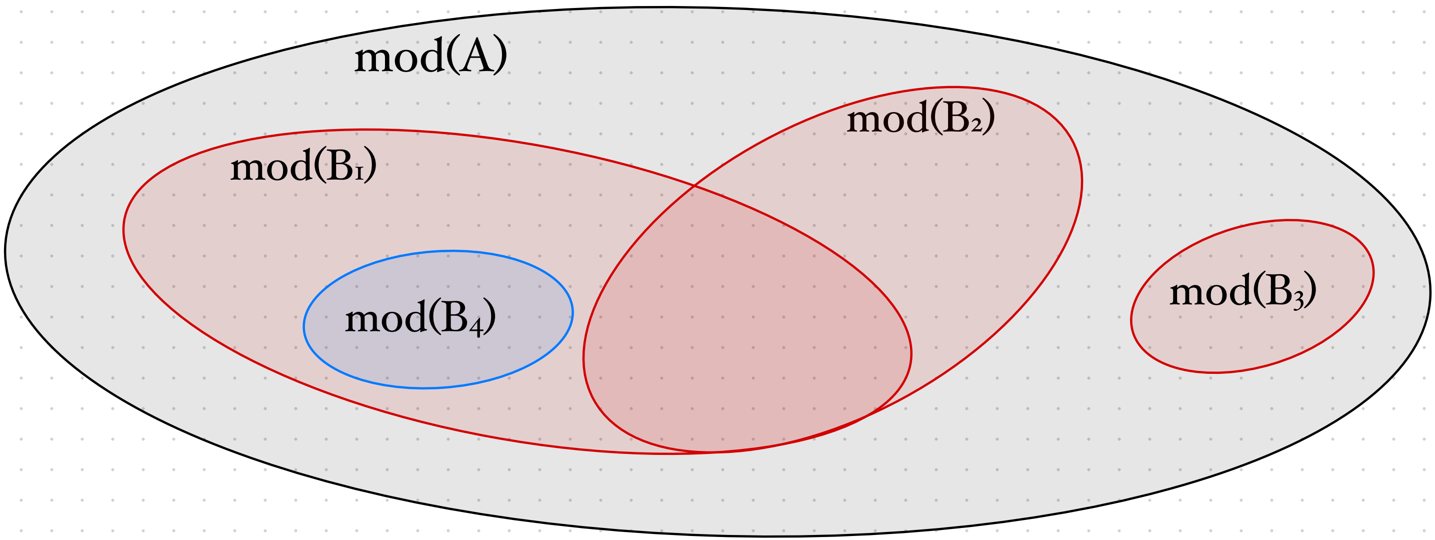

$W_i = \mathrm{mod}(B_i)$ are functorially-finite wide subcategories

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(M,P) $\in$ mod(A) x proj(A) **is τ-rigid if**

- Hom(M, τM) = 0,
- Hom(P, M) = 0.

Adachi-Iyama-Reiten, 2014

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τ-perpendicular wide subcategories

$M^{\perp} \cap {}^{\perp}\tau M \cap P^{\perp} \subseteq \mathrm{mod}(A)$

Adachi-Iyama-Reiten, 2014
Jasso, 2015

A category of wide subcategories of $\mathrm{mod}(A)$

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Definition The τ-cluster morphism category $\mathcal{T}(A)$ has:

- objects: τ-perpendicular wide subcategories
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Igusa-Todorov, 2017
Buan-Marsh, 2021
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Problem

$$\tau\text{-rigid}(A) \cap \text{mod}(A/I) \subsetneq \tau\text{-rigid}(A/I)$$
$$\text{wide}(A) \cap \text{mod}(A/I) \subsetneq \text{wide}(A/I)$$

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Observation:

$$\mathrm{tors}(A) \cap \mathrm{mod}(A/I) = \mathrm{tors}(A/I)$$

Igusa-Todorov, 2017
Buan-Marsh, 2021
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Wide subcategories as intervals of $\mathrm{tors}(A)$

Demonet-Iyama-Reading-
Reiten-Thomas, 2023
Asai-Pfeifer, 2022

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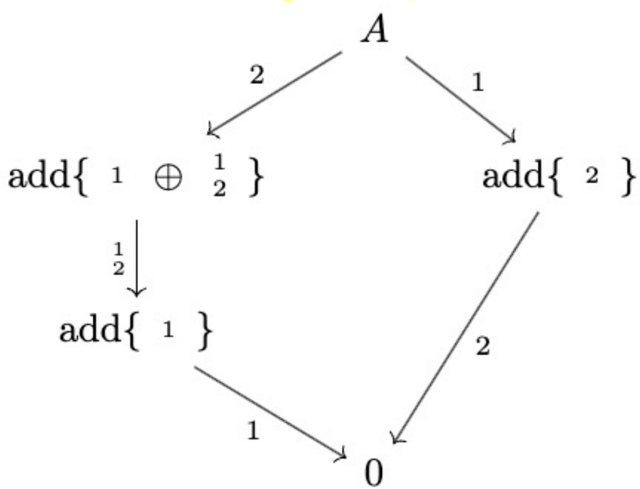

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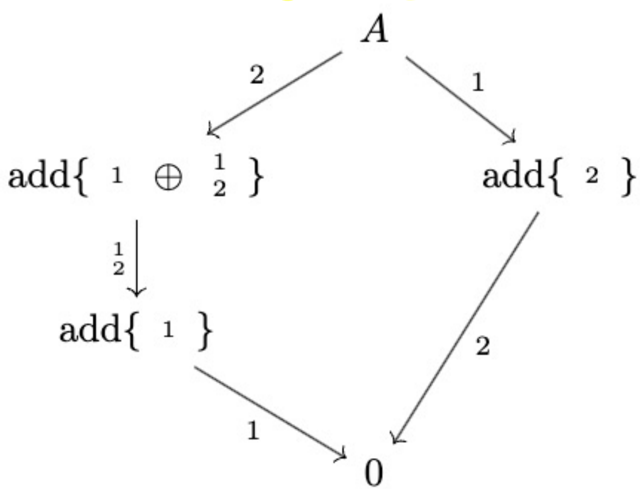

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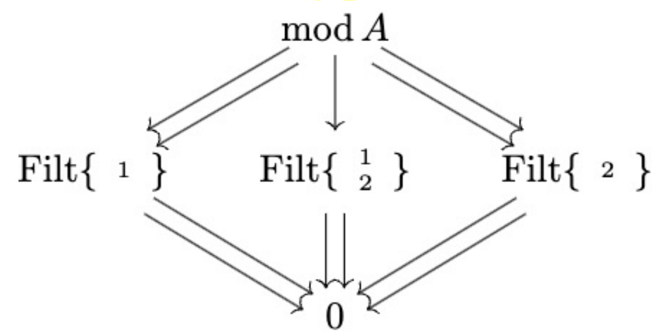

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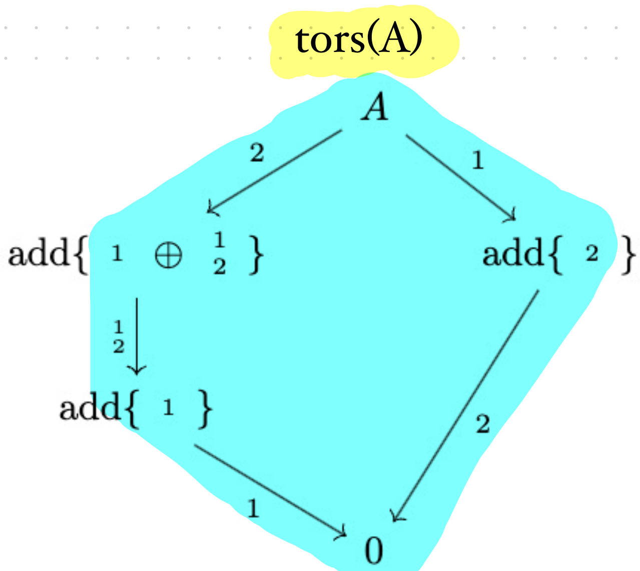

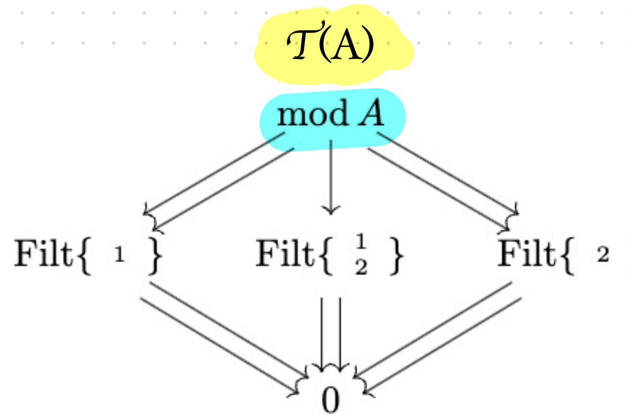

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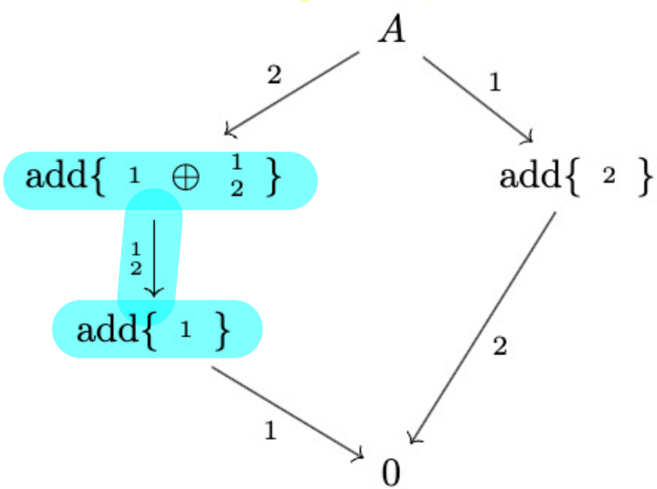

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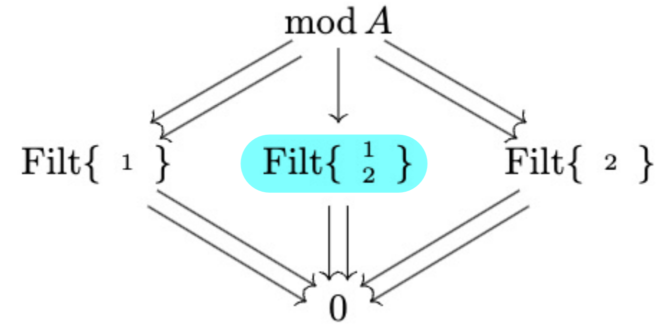

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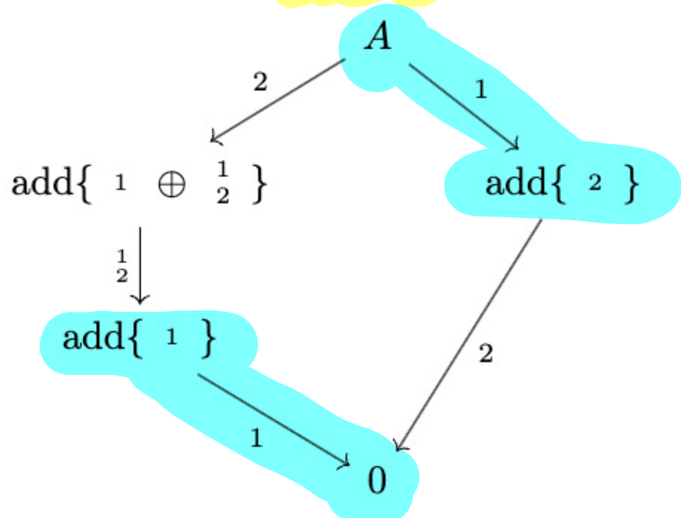

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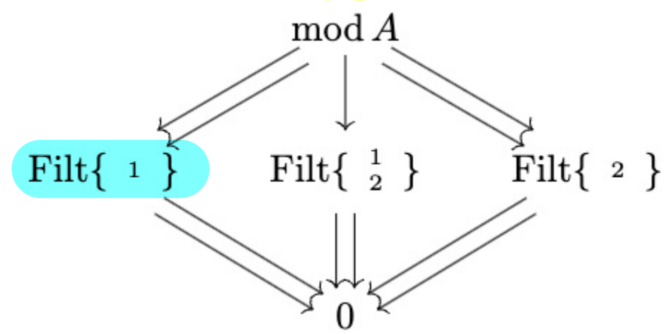

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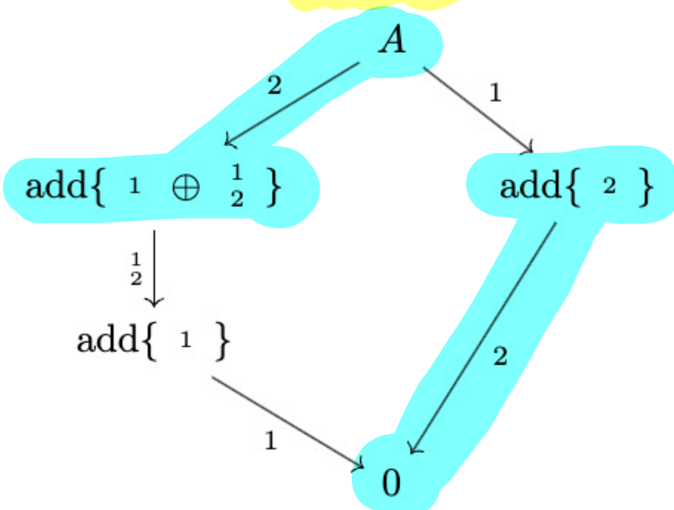

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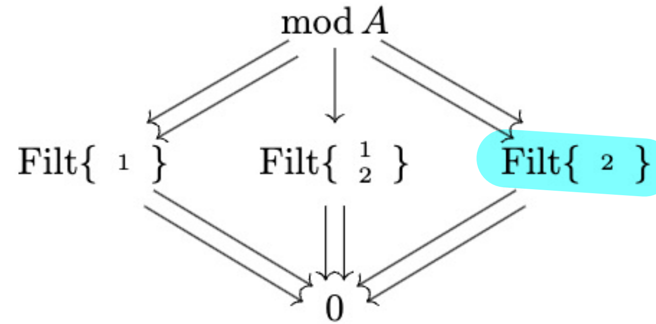

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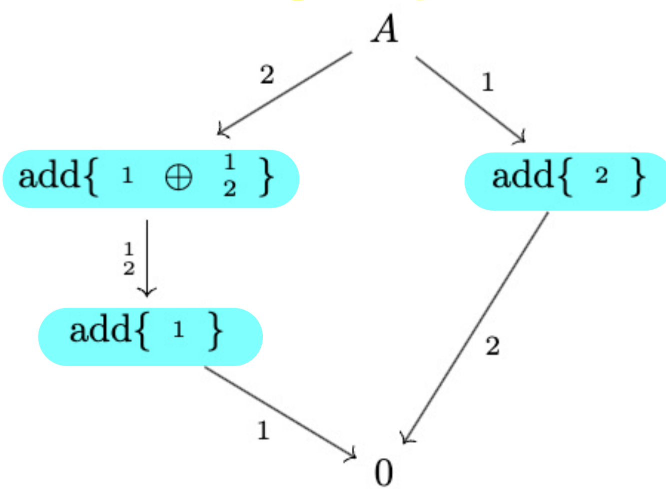

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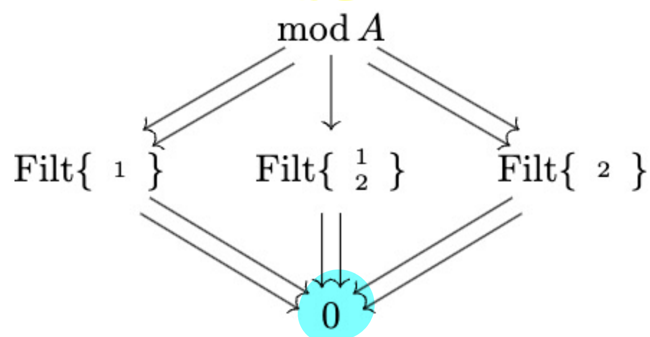

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Theorem If $\operatorname{tors}(A)$ is finite, then $\mathcal{F}$ is surjective-on-objects and there exists a faithful functor

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Thank you!

谢谢